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s-PROCESS STUDIES: EXACT SOLUTION TO A CHAIN HAVING TWO DISTINCT CROSS-SECTION VALUES

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ABSTRACT

We present the exact solution to an s-process chain having two distinct values for the cross-sections. The comparison of this exact solution with the CFHZ approximate solution to the same problem reveals that approximate solution to be adequate for most astrophysical problems.

Subject heading: nucleosynthesis

The numerical difficulty in evaluating the exact Bateman solution for large k of the s-process network of differential equations led Clayton *et al.* (1961, hereafter referred to as CFHZ) to develop an approximate solution (see also Clayton 1968). Unfortunately the difficulty of evaluating the exact solution not only limits its own usefulness but also makes it hard to test thoroughly the validity of the CFHZ approximate solution. Our present purpose is to present an exact solution to a simplified but realistic s-process problem. This exact solution is easily evaluated and compared with the CFHZ approximate solution to the same problem. The close agreement confirms the high accuracy of the CFHZ approximation.

One simplified problem that is not very realistic but is easily evaluated is the case where all neutron-capture cross-sections have the same value. The functional form of this solution provided the basis for the CFHZ approximation for $\Psi_k(\tau)$, defined as the product $\sigma_k N_k(\tau)$ per initial seed nucleus as a function of the neutron irradiation τ:

$$\Psi_k(\tau) \simeq \lambda_k \frac{(\lambda_k \tau)^{m_k - 1}}{\Gamma(m_k)} \exp(-\lambda_k \tau) \tag{1}$$

where the parameter λ_k replaces the constant value of the cross-section and the parameter m_k replaces the numbering index k. The values of the two parameters for each k are chosen in the CFHZ approximation by the requirement that the first three moments of $\Psi_k(\tau)$ are the same in the approximate and exact solutions.

We here consider the case where the cross-sections can assume either of two fixed values σ_1 or σ_2, and we will present the exact solution to this problem. Its form is only slightly more complicated than that of the solution for all cross-sections being equal; however, it is the solution to a much more realistic problem, because the s-process is dominated by the presence of several small cross-sections among many large ones. Therefore, the comparison of the CFHZ approximation for this problem with the exact solution to this problem provides a fair test of the accuracy of the CFHZ approximation to the real problem.

Before presenting the solution for two classes of cross-sections, we note that it can be easily enlarged to include a third class of infinitely large cross-sections. This generalization, though trivial mathematically, is also of physical relevance, because the exact values of the large cross-sections are relatively unimportant as far as the solution for Ψ_k is concerned. (They are, of course, important for the values of N_k.) In fact, a good measure of the importance of the value of the kth cross-section is the value of $1/\sigma_k$. The generalization to include infinite cross-sections comes about because when $\sigma_k \to \infty$ the differential equations immediately show that $\Psi_k(\tau) \to \Psi_{k-1}(\tau)$ for all τ, and the network effectively collapses to include only those species having finite (i.e., not *very* large) cross-sections. The value of Ψ_k for species with infinite (very large) cross-sections can be trivially inserted after solution of the simplified network. It would therefore be possible to replace the actual s-process chain by a physical approximation in which each cross-section σ_k had one of three values: σ_1 if σ_k is deemed small, σ_2 if σ_k is deemed intermediate, and ∞ if σ_k is deemed large. In figure 1 we show a histogram of the number of cross-sections having values in the indicated ranges, and one can see that the approximate sorting of cross-sections into one of three classes is plausible. Nonetheless, we will not pursue this method of approximating the real problem by the restricted one here, but will instead present the exact solution to the restricted problem and compare it to the CFHZ approximation to that problem. For the numerical comparisons, we will, however, use $\sigma_1 = 7$ mb and $\sigma_2 = 35$ mb as characteristic of the small and intermediate groups.

Suppose then that n_1 of the first k species have cross-sections equal to σ_1 and that n_2 have the value σ_2. Because we assume only these two distinct values for the cross-sections we have $k = n_1 + n_2$. Then the Laplace transform $\bar{\Psi}_k(s)$ of the solution $\Psi_k(\tau)$ for the problem in which the seed nuclei are initially at $k = 1$ is (CFHZ, Clayton 1968)

$$\bar{\Psi}_k(s) = \prod_{j=1}^{k} \frac{\sigma_j}{s + \sigma_j} = \left(\frac{\sigma_1}{s + \sigma_1}\right)^{n_1} \left(\frac{\sigma_2}{s + \sigma_2}\right)^{n_2}. \tag{2}$$

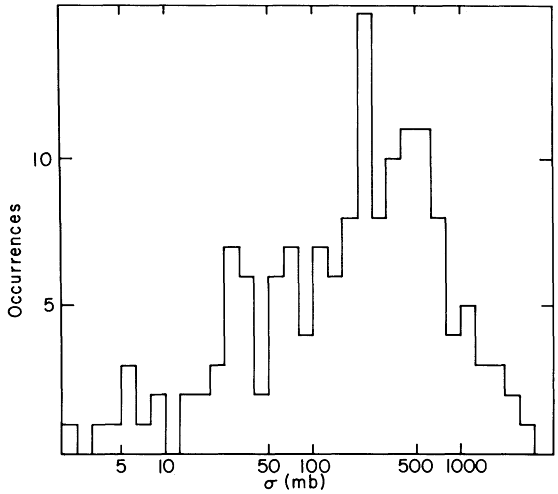

FIG. 1.—Histogram of the number of nuclei having $A \geq 56$ in the s-process chain whose cross-sections (in millibarns) lie in the indicated logarithmic interval. The values of the cross-sections are from Allen, Gibbons, and Macklin (1971). This chain could be approximated by a group of small cross-sections, a group of intermediate cross-sections, and a group of infinite cross-sections.

The solution is independent of the detailed locations within the chain of the cross-sections σ_1 and σ_2. For the inversion of this transform by contour integration, one sees that the integrand has a pole of order n_1 at $s = -\sigma_1$ and a pole of order n_2 at $s = -\sigma_2$. By applying Leibnitz's rule for the derivative of a product in the evaluation of the residues, we obtain the result

$$\Psi_k(\tau) = \sigma_1 \frac{(\sigma_1 \tau)^{n_1-1}}{(n_1-1)!} e^{-\sigma_1 \tau} \left(\frac{\sigma_2}{\sigma_2-\sigma_1}\right)^{n_2} \sum_{j=0}^{n_1-1} \frac{(n_2-1+j)!\,(n_1-1)!}{(n_2-1)!\,(n_1-1-j)!\,j!} \left[\frac{-1}{(\sigma_2-\sigma_1)\tau}\right]^j$$
$$+ \sigma_2 \frac{(\sigma_2 \tau)^{n_2-1}}{(n_2-1)!} e^{-\sigma_2 \tau} \left(\frac{\sigma_1}{\sigma_1-\sigma_2}\right)^{n_1} \sum_{j=0}^{n_2-1} \frac{(n_1-1+j)!\,(n_2-1)!}{(n_1-1)!\,(n_2-1-j)!\,j!} \left[\frac{-1}{(\sigma_1-\sigma_2)\tau}\right]^j . \quad (3)$$

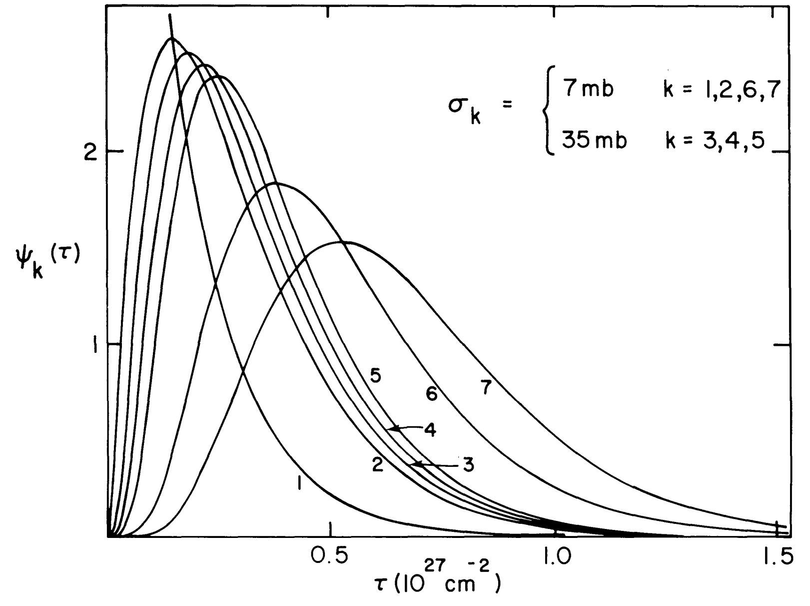

FIG. 2.—Exact display of $\Psi_k(\tau)$ for $k = 1$ through 7 of a simple network. Two small cross-sections (7 mb) are followed by three larger ones (35 mb) and then two more small ones. The Poisson-like shape allows Ψ_k to be approximated well by eq. (1). The curves representing species with large cross-sections ($k = 3, 4, 5$) track their predecessors closely. This assignment of cross-section values is not intended to simulate the real problem.

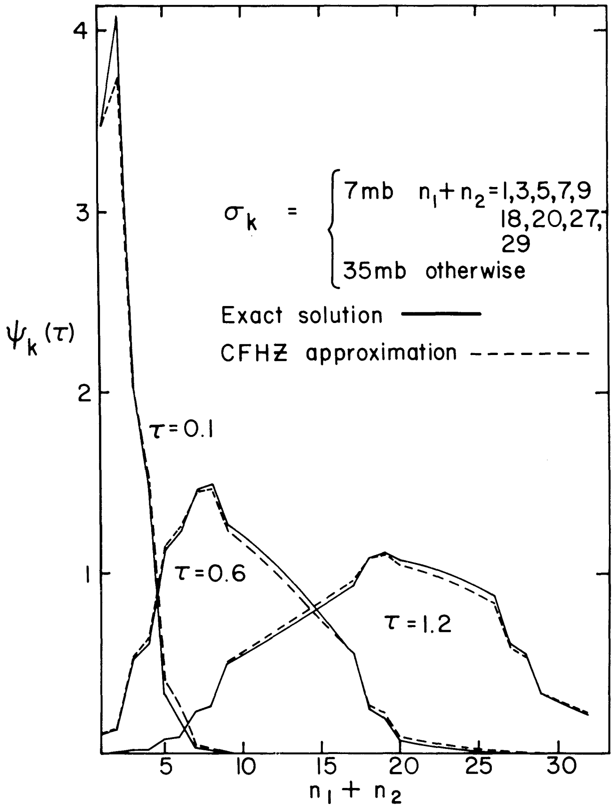

FIG. 3.—Linear plot of $\Psi_k(\tau)$ for a chain resembling the real problem. The value $\sigma_1 = 7$ mb was assigned to all k having $\sigma_k \leq 15$ mb, and the value $\sigma_2 = 35$ mb was assigned to those k having 15 mb $< \sigma_k \leq 65$ mb. The value $\sigma_3 = \infty$ was assigned to all species having $\sigma_k > 65$ mb, whereupon these species were deleted from the network as being superfluous. The numbering index k is then taken to be $n_1 + n_2$. The solid curves are the exact solution, and the dashed curves are the CFHZ approximation to the same problem.

This rather formidable looking expression is actually fairly simple. It is the sum of two terms symmetric under interchange of the labels 1 and 2, a fact convenient for computation. The first term is just the one-class solution in σ_1 and n_1 multiplied by the n_2 power of a ratio of cross-sections times a polynomial of order $n_1 - 1$ in $[(\sigma_2 - \sigma_1)\tau]^{-1}$. The terms of the polynomial alternate in sign, and its leading term is unity. The second term is similar except for the alternation of sign of the polynomial, but with the labels interchanged. The behavior may be dominated by the first term if $\sigma_2 \gg \sigma_1$ and if τ is not too small, in which case the exponential in the second term renders it small in comparison with the first term and, in the same circumstances, the argument of the polynomial becomes a small number so that only its first few terms are important. In this case $\sigma_2/(\sigma_2 - \sigma_1)$ is only slightly larger than unity, and the polynomial is slightly smaller than unity. Then the behavior of the solution is dominated by the one-class solution, and its character as a function of k is determined by the positions at which the small cross-sections enter the network, i.e., by when n_1 is incremented.

This point is further illustrated by figure 2, which shows the behavior of $\Psi_k(\tau)$ for the first seven values of k for a simple test network. The falling exponential represents the seed nucleus, which is destroyed but not replenished. The other curves are of roughly Poisson shape, which makes us hopeful for the fortunes of the CFHZ approximation, based as it is on just such a functional form. Each such curve reaches its single maximum as it crosses the preceding curve, as it must if it satisfies the differential equations. The curves 3, 4, and 5, resulting from the addition of species with large cross-sections, differ but slightly from the previous curve 2 and from each other, at least as compared with the large differences evident in curves 6 and 7, representing as they do the addition of small cross-sections.

The shape of $\Psi_k(\tau)$ at constant τ for a network chosen to simulate the physical problem is shown in figure 3. The distribution is sharply peaked near the seed for small exposures, propagating toward large k as τ increases, and spreading as it goes due to the random nature of the captures. The slope of each curve is considerably steeper where small cross-sections are being added than where large ones enter, and would of course become flat were we to insert species with infinite cross-sections. Also in figure 3 we display the CFHZ approximation for the same

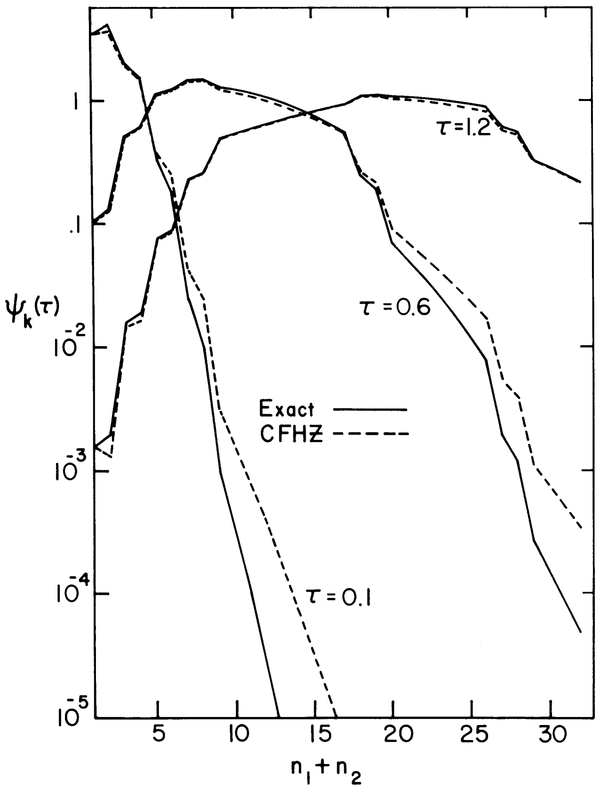

FIG. 4.—The same curves as fig. 3 redrawn semilogarithmically to display the behavior for small values of Ψ. The CFHZ approximation overestimates Ψ_k for k well beyond the peak, but the absolute error is small.

problem. The functional form is that of equation (1), but the prescription for the parameters becomes simply (CFHZ)

$$\lambda_k = \frac{n_1/\sigma_1 + n_2/\sigma_2}{n_1/\sigma_1{}^2 + n_2/\sigma_2{}^2}, \tag{4}$$

$$m_k = \frac{(n_1/\sigma_1 + n_2/\sigma_2)^2}{n_1/\sigma_1{}^2 + n_2/\sigma_2{}^2}. \tag{5}$$

Figure 3 shows that the CFHZ approximation matches the exact solution quite well. The relative error is better displayed in figure 4, which is the same set of curves redrawn semilogarithmically. There we see that the relative error can be quite large at given k for τ small enough that the peak of the distribution has not yet reached that value of k. The CFHZ approximation then overestimates the value of Ψ_k, as figure 4 shows most clearly. But in such regions the value of Ψ_k is itself quite small, so that the absolute error is everywhere small. This evidence of the general excellence of the CFHZ approximation is our major result. It can probably be used in all astrophysical applications.

We had considered that this new exact solution might lead to an improved approximation for the physical problem by choosing the four parameters of the two-class solution to match moments of the exact solution of the physical problem in a straightforward extension of the method of generating the CFHZ approximation from the one-class solution. We have tried several approaches of this type and find the mathematics sufficiently messy that one advantage of an approximation—ease of use—is lost. We did find some improved approximations while still matching only the first three moments, and further improvements could no doubt be obtained by matching even more moments. In addition, an approximation designed for small τ could be generated in this manner. However, it appears that little would be gained by such programs, for the CFHZ approximation seems to be quite good in most regions of astrophysical interest.

One possible use of this exact solution is to estimate the percentage error in the wings of Ψ_k, as for example in figure 4, and to apply these as known corrections to the CFHZ approximation. The procedure as we see it would be to choose the values of σ_1 and σ_2 and their positions in the chain, and to choose where the infinite cross-sections should be inserted, so that the CFHZ approximate solution to this problem approximates well the CFHZ approximate solution to the real problem. The artificial problem could then be said to simulate the real one. Then the exact solution to the artificial problem could be compared with the CFHZ solution, as we have done here, to generate realistic correction factors $\Delta_k(\tau) \equiv \Psi_k(\tau\colon \text{exact})/\Psi_k(\tau\colon \text{CFHZ})$. These correction factors could then be confidently applied to the CFHZ approximation to the real problem. At this time we find figure 4 to be adequate.

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